Section 6.2

Monday, April 13, 2020 2:36 PM

Integrals of rational functions

Recall that a rational function is a function of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. We will first learn how to integrate rational functions $\frac{R(x)}{Q(x)}$ where $\deg(R(x)) < \deg(Q(x))$.

Given a rational function $\frac{R(x)}{Q(x)}$ where $\deg(R(x)) < \deg(Q(x))$, we first write $Q(x)$ as

$$Q(x) = K(x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_n)^{k_n}(x^2+b_1x+c_1)^{\ell_1}(x^2+b_2x+c_2)^{\ell_2}\cdots(x^2+b_mx+c_m)^{\ell_m}$$

where the quadratic terms $x^2+b_ix+c_i$ have no real roots. In this case, we can write $\frac{R(x)}{Q(x)}$ as

$$\frac{R(x)}{Q(x)} = \frac{A_1^1}{(x-a_1)^1} + \frac{A_2^1}{(x-a_1)^2} + \cdots + \frac{A_{k_1}^1}{(x-a_1)^{k_1}} + \frac{A_1^2}{(x-a_2)^1} + \frac{A_1^2}{(x-a_2)^2} + \cdots + \frac{A_{k_2}^2}{(x-a_2)^{k_2}}$$

$$+ \cdots + \frac{B_1^1x+C_1^1}{(x^2+b_1x+c_1)^1} + \frac{B_2^1x+C_2^1}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{\ell_1}^1x+C_{\ell_1}^1}{(x^2+b_1x+c_1)^{\ell_1}}$$

$$+ \frac{B_1^2x+C_1^2}{(x^2+b_2x+c_2)^1} + \frac{B_2^2x+C_2^2}{(x^2+b_2x+c_2)^2} + \cdots + \frac{B_{\ell_2}^2x+C_{\ell_2}^2}{(x^2+b_1x+c_2)^{\ell_2}} + \cdots$$

This sum is called **the partial fraction decomposition** of $\frac{R(x)}{Q(x)}$.

Examples:

$$\frac{2x+3}{x^3+2x^2+x} = \frac{2x+3}{x(x^2+2x+1)} = \frac{2x+3}{x(x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{(x+1)^1} + \frac{C}{(x+1)^2}$$

$$\frac{5x^2+4}{x^6+2x^4+x^2} = \frac{5x^2+4}{x^2(x^4+2x^2+1)} = \frac{5x^2+4}{x^2(x^2+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+1)} + \frac{Ex+F}{(x^2+1)^2}$$

After finding the partial fraction decomposition of $\frac{R(x)}{Q(x)}$, we integrate each term separately, in order to integrate $\frac{R(x)}{Q(x)}$. Note that these terms are easier to integrate.

Example:

- $\int \frac{3}{x-2}\,dx = \ln|x-2| + C$

- $$\int \frac{7}{(x-3)^5}\,dx = \frac{7(x-3)^{-4}}{4} + C$$

- $$\int \frac{4x+1}{4x^2+4x+3}\,dx = \int \frac{4x+1}{4\left(x^2+x+\frac{3}{4}\right)}\,dx$$

$$= \frac{1}{4}\int \frac{4x+1}{\underbrace{\left(x+\frac{1}{2}\right)^2}_{x^2+x+\frac{1}{4}}+\frac{1}{2}}\,dx = \frac{1}{4}\left(\int \frac{4x+2}{\left(x+\frac{1}{2}\right)^2+\frac{1}{2}}\,dx - \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{2}}\,dx\right)$$

$$\boxed{\begin{array}{l}\left(x+\frac{1}{2}\right)^2+\frac{1}{2}=u \\ (2x+1)\,dx = du\end{array}}$$

$$= \frac{1}{4}\int \frac{2\,du}{u} - \frac{1}{4}\int \frac{1}{\frac{1}{2}\left(2\left(x+\frac{1}{2}\right)^2+1\right)}\,dx$$

$$= \frac{1}{2}\ln|u| - \frac{1}{4}\int \frac{2\,dx}{\left(\sqrt{2}x+\frac{\sqrt{2}}{2}\right)^2+1}$$

$\sqrt{2}x+\frac{\sqrt{2}}{2} = v$

$\sqrt{2}\,dx = dv$

$$= \frac{1}{2}\ln|u| - \frac{1}{4}\int \frac{\sqrt{2}\,dv}{v^2+1}$$

$$= \frac{1}{2}\ln|u| - \frac{\sqrt{2}}{4}\arctan(v) + C$$

$$= \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\right| - \frac{\sqrt{2}}{4}\arctan\left(\sqrt{2}x+\frac{\sqrt{2}}{2}\right) + C$$

- $$\int \frac{1}{(1+x^2)^3}\,dx$$ $\rightsquigarrow$ requires some inverse trig substitutions that we'll learn later

Given a rational function $\frac{R(x)}{Q(x)}$, if $\deg(R(x)) \geq \deg(Q(x))$, then we carry out a "long division for polynomials" to write $\frac{R(x)}{Q(x)}$ as

$$\frac{R(x)}{Q(x)} = P(x) + \frac{S(x)}{Q(x)}$$ where $\deg(S(x)) < \deg(Q(x))$ in which case

we can apply the previous techniques

Example: Find $$\int \frac{2x^5 - x^4 - 30x + 17}{x^4 - 16}\,dx$$

Solution: We have that

$$\int \frac{2x^5-x^4-30x+17}{x^4-16}\,dx =$$

$$\begin{array}{l|l} \cancel{2x^5}-x^4-30x+17 & x^4-16 \\ \hline -\;\cancel{2x^5}-32x & 2x-1 \\ \hline \cancel{-x^4}+2x+17 & \\ -\;\cancel{-x^4}+16 & \\ \hline 2x+1 & \end{array}$$

$$\int 2x-1+\left(\frac{2x+1}{x^4-16}\right)dx =$$

$$\int 2x-1\,dx+\int \frac{2x+1}{x^4-16}\,dx =$$

$$x^2-x+\int \frac{\frac{5}{32}}{x-2}+\frac{\frac{3}{32}}{x+2}+\frac{\frac{-2}{8}x+\frac{-1}{8}}{x^2+4}\,dx$$

$$= x^2-x+\frac{5}{32}\ln|x-2|+\frac{3}{32}\ln|x+2|$$

$$+\int \frac{-2}{8}\frac{x}{x^2+4}+\frac{-1}{8}\frac{1}{x^2+4}\,dx$$

$$= x^2-x+\frac{5}{32}\ln|x-2|+\frac{3}{32}\ln|x+2|$$

$$+\int \frac{-1}{4}\frac{x}{x^2+4}\,dx+\int \frac{-1}{8}\frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)}\,dx$$

$\boxed{x^2+4=u}$ $\boxed{\frac{x}{2}=v}$

$$= x^2-x+\frac{5}{32}\ln|x-2|+\frac{3}{32}\ln|x+2|$$

$$+\int \frac{-1}{4}\frac{1}{2}\frac{du}{u}+\int \frac{-1}{8}\frac{1}{2}\frac{dv}{v^2+1}$$

$$= x^2-x+\frac{5}{32}\ln|x-2|+\frac{3}{32}\ln|x+2|-\frac{1}{8}\ln|x^2+4|-\frac{1}{16}\arctan\left(\frac{x}{2}\right)+C$$

$$\frac{2x+1}{x^4-16}=\frac{2x+1}{(x^2-4)(x^2+4)}$$

$$=\frac{2x+1}{(x-2)^1(x+2)^1(x^2+4)^1}$$

$$=\frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$$

$$=\frac{A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x-2)(x+2)}{(x-2)(x+2)(x^2+4)}$$

$$\Downarrow$$

$$A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x^2-4)=2x+1$$

Plug in $x=2$, so $A\cdot 32=5 \Rightarrow \boxed{A=\frac{5}{32}}$

Plug in $x=-2$, so $B\cdot(-32)=-3 \Rightarrow \boxed{B=\frac{3}{32}}$

Plug in $x=0$, so $8A-8B-4D=1 \Rightarrow$

$8\cdot\frac{2}{32}-4D=1 \Rightarrow \boxed{D=\frac{-1}{8}}$

Plug in $x=1$ and get $\boxed{C=\frac{-2}{8}}$

Example: Find $\int \frac{x^2+2}{4x^5+4x^3+x}\,dx$

Solution:

$$\frac{x^2+2}{4x^5+4x^3+x} = \frac{x^2+2}{x(4x^4+4x^2+1)} = \frac{(x^2+2)}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(2x^2+1)} + \frac{Dx+E}{(2x^2+1)^2}$$

$$\left.\right\} \quad \begin{aligned} A &= 2 \\ B &= -4 \\ C &= 0 \\ D &= -3 \\ E &= 0 \end{aligned}$$

Check this!

$$\int \frac{x^2+2}{4x^5+4x^3+x}\,dx = \int \frac{2}{x} - \frac{4x}{2x^2+1} - \frac{3x}{(2x^2+1)^2}\,dx$$

$$= 2\ln|x| - \int \frac{4x\,dx}{2x^2+1} - \int \frac{3x}{(2x^2+1)^2}\,dx$$

$$\boxed{\begin{aligned} 2x^2+1 &= u \\ 4x\,dx &= du \end{aligned}}$$

$$= 2\ln|x| - \int \frac{du}{u} - \int \frac{3}{4}\,\frac{du}{u^2} = 2\ln|x| - \ln|2x^2+1| + \frac{3}{4}\,\frac{1}{2x^2+1} + C$$